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Examples of strongly π-regular rings

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Abstract

Our study in this note is concentrated on extending the class of strongly π-regular rings, observing the structures of them. We call a ring *locally finite* if every finite subset in it generates a finite semigroup multiplicatively. We first study the structures of locally finite rings and then study relations between locally finite rings and other related rings. We also study the strong π-regularity of some kinds of semiperfect rings with nil Jacobson radicals.

MSC: 16D25; 16N40; 16S36

1. Introduction

The strong π-regularity has roles in module theory and ring theory as we see in Ara [2], Azumaya [5], Birkenmeier et al. [8], Burgess and Menal [10], Hirano [18], Nicholson [22], Rowen [24,25], and so on. Our study in this note is concentrated on extending the class of strongly π-regular rings, observing the structures of them.

In Section 2, we are motivated by the following Jacobson's well-known result: Let R be a ring in which for every $a \in R$ there exists an integer $n(a) > 1$, depending on a, such that $a^{n(a)} = a$ then R is commutative. We therefore consider a generalized situation to the noncommutativity. We shall call a ring *locally finite* if every finite subset in it generates a finite semigroup multiplicatively. We first show the following: (i) each finite subset of a locally finite ring generates a finite subring; (ii) If R/I and

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I are both locally finite for some proper ideal I in a ring R then so is R, where I is considered as a subring of R without identity; (iii) the local finiteness is a Morita invariant property; (iv) for a locally finite ring R, R is abelian semiprimitive if and only if R is commutative regular; (v) for a right Goldie locally finite ring R, R is semiprime if and only if R is a finite direct product of full matrix rings over locally finite fields. It is well-known that the 2-primal condition helps to simplify other ring conditions, and so it may be somewhat meaningful to study relations between the local finiteness and the 2-primal condition. We actually show that locally finite abelian rings are 2-primal when they are of bounded index of nilpotency, finding a related counterexample.

Rowen [25, Example 2.4] found semiperfect rings with nil Jacobson radical but not strongly π-regular; hence it may be interesting to find some criteria on Jacobson radicals which assure that semiperfect rings may be strongly π-regular, by the argument after [25, Example 2.5]. In Section 3, we study the strong π-regularity of some kinds of semiperfect rings with nil Jacobson radical. Local rings are strongly π-regular when Jacobson radicals are nil, so we observe the generalized situations of this case. If R is a right continuous ring then $R/J(R)$ is von Neumann regular by Utumi [27, Theorem 4.6], so it may be useful to find conditions for which right continuous rings may be strongly π-regular. In fact we obtain the following comparable results: (i) Let R be a right continuous ring with $J(R)$ nil. If $J(R)$ is not essential in R as a right ideal and $R/J(R)$ is a direct product of two division rings, then R is strongly π-regular. (ii) Let R be a right continuous ring with $J(R)$ nil. If every essential right ideal of R is 2-sided and $R/J(R)$ is a direct product of two division rings, then R is strongly π-regular.

Throughout this paper all rings are associative with identity unless other conditions are given. Given a ring R, the Jacobson radical, the prime radical and the set of all nilpotent elements are denoted by $J(R)$, $P(R)$ and $N(R)$, respectively. Any finite ring is locally finite obviously, and the direct sum of locally finite rings, with an identity appended if necessary, may be also locally finite by simple direct computations. A ring R is called *strongly π-regular* if for every a in R there exist a positive integer n, depending on a, and an element b in R satisfying $a^n = a^{n+1}b$. It is obvious that a ring R is strongly π-regular if and only if R satisfies the descending chain condition on principal right ideals of the form $aR \supseteq a^2R \supseteq \cdots$, for every a in R. A ring R is called *π-regular* if for each $a \in R$ there exist a positive integer n, depending on a, and $b \in R$ such that $a^n = a^nba^n$, Strongly π-regular rings are π-regular by Azumaya [5], and it is easy to show that the Jacobson radicals of π-regular rings are nil. Dischinger [13] showed that the strongly π-regularity is left-right symmetric. For a division ring D and a right D-module V, notice that the endomorphism ring of V over D is strongly π-regular if and only if V is finite dimensional over D.

2. Locally finite rings

In this section, we first show that the local finiteness is a Morita invariant property and locally finite rings are strongly π-regular. Next we study the properties of locally finite rings, obtaining some connections between locally finite rings and some kinds of manageable rings. The $n \times n$ full matrix ring over a finite ring S, say R_n, is clearly locally finite for any positive integer n because it is finite. As an infinite case, the $Z(S)$-subalgebra of T generated by $\oplus_{n=1}^{\infty} R_n$ (the direct sum of R_n's) and 1_T is locally finite, where $Z(S)$ is the center of S, $T = \prod_{n=1}^{\infty} R_n$ (the direct product of R_n's), and 1_T is the identity of T.

Proposition 2.1. *Let R be a ring.*

(1) *If R is locally finite, then so is every factor ring (hence every homomorphic image) of R.*
(2) *Subrings (not necessarily with identity) of locally finite rings are locally finite.*
(3) *The direct sum of locally finite rings, with an identity appended if necessary, is locally finite.*
(4) *The direct limit of a direct system of locally finite rings, with an identity appended if necessary, is locally finite.*

Proof. The proofs of (1), (2) and (3) are obtained by direct computations.

(4) Note that a direct limit of a direct system of rings may be a factor ring of the direct sum of the rings, and so we have the result by (1) and (3). □

Example 1. (1) Based on Proposition 2.1(3), one may conjecture that each direct product of locally finite rings is also locally finite. However that is not valid for infinite direct product as follows. Let $R_n = \mathbb{Z}_{3^n}$ with n a positive integer and $\mathbb{Z}_{3^n}$ the ring of integers modulo 3^n. Next put $R = \prod_{n=1}^{\infty} R_n$, and consider $a = (a_n) \in R$ with $a_n = 2$ for all n. Then every R_n is finite, but R is not locally finite since a generates an infinite multiplicative semigroup.

(2) The following is an example of direct limit. Let K be any finite field and $M_t(K)$ be the $t \times t$ full matrix ring over K. For each positive integer i consider the standard unital embedding of matrix rings $\phi_i : M_{2^i}(K) \to M_{2^{i+1}}(K)$, and let R be the direct limit of the direct system $(M_{2^i}(K), \phi_i)$. Then R is locally finite by Proposition 2.1(4). □

Theorem 2.2. *Let R be a ring.*

(1) *R is locally finite if and only if each finite subset of R generates a finite subring (not necessarily with identity).*
(2) *If R/I and I are both locally finite for some proper ideal I in R then so is R, where I is considered as a subring of R without identity.*
(3) *Suppose that R is locally finite. Then for every finitely generated projective right R-module P, $End_R(P)$ is locally finite; especially the local finiteness is a Morita invariant property, where $End_R(P)$ is the endomorphism ring of P over R.*

Proof. (1) It suffices to show the necessity. First notice that the characteristic of R is nonzero. For, assuming that the characteristic of R is zero, then $2 = 1 + 1$ forms an infinite multiplicative semigroup, a contradiction. Next let A be a finite subset of R and B be the subring generated by A. The multiplicative semigroup (A) generated by A is finite since R is locally finite, say $(A) = \{a_1, a_2, \ldots, a_n\}$. Notice that each element of B is of the form $k_1a_1 + k_2a_2 + \cdots + k_na_n$ for some integers k_i. But the characteristic of R is nonzero by the previous argument, say h; then we may set $0 \leqslant k_i \leqslant h-1$ for all i. So the cardinality of B is equal to or less than h^n, implying that B is also finite.

(2) Write $\bar{r}$ in place of $r + I \in R/I$. First we show that the characteristic of R is nonzero. Since R/I is locally finite, the characteristic of R/I is nonzero, say m, by the same manner as in the proof of (1). I is also locally finite, so $(m1)^n = m^n 1 = 0$ for some positive integer n; hence R has nonzero characteristic. Then we conclude also as in the proof of (1) that every finite subset of I generates a (nonunital) finite subring of I. Next let $A = \{a_1, a_2, \ldots, a_s\}$ be a finite subset of R. Then $\bar{A} = \{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_s\}$ generates a finite subring of R/I by (1) since R/I is locally finite, say

$$\bar{B} = \{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_s, \bar{b}_{s+1}, \bar{b}_{s+2}, \ldots, \bar{b}_t\}$$

for some $t \geqslant s$ and suitable $b_g \in R$ (if exist) with $s+1 \leqslant g \leqslant t$, where $b_h = a_h$ for $h = 1, 2, \ldots, s$. Set $C = \{b_1, b_2, \ldots, b_t\}$. Notice that for any $b_p, b_q \in C$, $b_pb_q = b_v + \alpha$ for some $b_v \in C$ and $\alpha \in I$. Let L be the set of all such α's in this situation. Then L is finite since $\{b_pb_q \mid b_p, b_q \in C\}$ is finite and $\bar{b}_u$'s are distinct each other. Next let J be the set of all elements of the following forms

$$\alpha, x\alpha, \alpha y \quad \text{and} \quad x\alpha y,$$

where $x, y \in C$ with and $\alpha \in L$. Then J is a finite subset of I because C and L are finite, whence J generates a finite subring of I by the argument above, say K. In the following computation, we will use freely the fact that K is a ring. By the argument above, we already have that for any $a_i, a_j \in C$, $a_ia_j = b_v + \alpha$ for some $b_v \in C$ and $\alpha \in I$. Next for any $a_i, a_j, a_k \in A$, we have

$$a_ia_ja_k = (b_v + \alpha)a_k = b_va_k + \alpha a_k = b_w + \beta + \alpha a_k$$

for some $b_v, b_w \in C$ and $\alpha, \beta \in L$; but $\beta + \alpha a_k$ is also in K since β and αa_k are in J. For any $a_i, a_j, a_k, a_l \in A$, we have

$$\begin{aligned} a_ia_ja_ka_l &= ((b_v + \alpha)a_k)a_l = (b_va_k + \alpha a_k)a_l \\ &= (b_w + \beta + \alpha a_k)a_l = b_wa_l + \beta a_l + \alpha a_ka_l = b_d + \delta + \beta a_l + \alpha b_e + \alpha\zeta \end{aligned}$$

for some $b_v, b_w, b_d, b_e \in C$ and $\alpha, \beta, \delta, \zeta \in L$; but $\delta + \beta a_l + \alpha b_e + \alpha\zeta$ is also in K. Now inductively we obtain that for any subset $\{a_{i_1}, a_{i_2}, \ldots, a_{i_u}\}$ of A

$$a_{i_1}a_{i_2}\cdots a_{i_u} = c + \delta \quad \text{for some } c \in C \text{ and } \delta \in K.$$

Consequently A generates a finite multiplicative semigroup in R since C and K are both finite, proving that R is locally finite.

(3) We use $Mat_n(R)$ to denote the $n \times n$ full matrix ring over R for any positive integer n. eRe is locally finite for every $e = e^2 \in R$ by Proposition 2.1(2). Let T be a finite subset of $Mat_n(R)$ and let $S = \{a_{ij} \mid (a_{ij}) \in T\}$. Then S is finite since T is finite, and S generates a finite subring in R, say U, by (1). Notice that every matrix, which is in the multiplicative semigroup generated by T in $Mat_n(R)$, must have entries in U. Consequently the multiplicative semigroup generated by T is finite, and so $Mat_n(R)$ is locally finite for each positive integer n. Note that $End_R(P) \cong eMat_n(R)e$ for some

$e^2 = e \in Mat_n(R)$ and some positive integer n. Thus the proof is complete, applying Proposition 2.1(2). □

The converse of Theorem 2.2(2) already holds by Proposition 2.1(1) and (2). The characteristic of a locally finite ring is nonzero by the proof of Theorem 2.2(1), but there is a ring of nonzero characteristic but not locally finite. Let F be the field of integers modulo 2, $F[x]$ be the polynomial ring over F with x the indeterminate, and $R_n = F[x]/x^nF[x]$ for any positive integer n. Denote $\prod_{n=1}^{\infty} R_n$ by R, and consider the element $a = (\bar{1}, \bar{x}, \bar{x}, \ldots, \bar{x}, \ldots)$ in R with $\bar{x} = x + x^nF[x]$. Then a generates an infinite multiplicative semigroup of R, so the characteristic of R is 2 but R is not locally finite.

Corollary 2.3. *For a ring R and a positive integer n, the following conditions are equivalent*:

(1) *R is locally finite.*
(2) *The $n \times n$ full matrix ring over R is locally finite.*
(3) *The $n \times n$ upper triangular matrix ring over R is locally finite.*
(4) *The $n \times n$ lower triangular matrix ring over R is locally finite.*

Proof. (1) $\Rightarrow$ (2): By the proof of Theorem 2.2(3). (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (4) $\Rightarrow$ (1) are obtained from Proposition 2.1(2). □

Next we study the properties of locally finite rings.

Lemma 2.4. *Let R be a ring.*

(1) *If R is locally finite and $a \in R$, then a^t is an idempotent for some positive integer t.*
(2) *Locally finite rings are strongly π-regular.*
(3) *If R is locally finite then left and right singular ideals of R are contained in $J(R)$.*

Proof. (1) Since R is locally finite, $a^m = a^{m+n}$ for some integers $m, n \geqslant 1$. Then inductively we have $a^m = a^m a^n = a^m a^{2n} = \cdots = a^m a^{mn} = a^{m(n+1)}$; hence letting $s = n+1$ then we obtain $a^m = (a^m)^s$ with $s \geqslant 2$. But $a^{(s-1)m} = a^{(s-2)m} a^m = a^{(s-2)m}(a^m)^s = a^{2(s-1)m} = (a^{(s-1)m})^2$, so a^t is an idempotent in R with $t = (s-1)m = mn$.

(2) By (1), a locally finite ring R satisfies the descending chain condition on principal right ideals of the form $aR \supseteq a^2R \supseteq \cdots$, for every a in R; hence R is strongly π-regular.

(3) By (2), [10, Proposition 2.6(iii); 22, Proposition 1.9]. □

The converse of Lemma 2.4(2) is obviously not true in general for strongly π-regular rings of characteristic zero (e.g., the field of rationals). By Lemma 2.4(2), locally finite rings have the exchange property as regular modules by [10, Proposition 2.6(iii)] and [22, Proposition 1.8(1)].

A ring R is called *von Neumann regular* (simply *regular*) if for each $a \in R$ there exists $x \in R$ such that $a = axa$. Regular rings are clearly π-regular. A ring R is called *strongly regular* if for each $x \in R$ there exists $y \in R$ such that $x = x^2 y$. Strongly regular rings are strongly π-regular obviously. A ring is called *abelian* if every idempotent is central, and a ring is called *reduced* if it has no nonzero nilpotent elements. Reduced rings are abelian by simple computations. Note that a ring R is strongly regular if and only if R is abelian regular if and only if R is reduced regular, by Goodearl [14, Theorems 3.2 and 3.5].

Proposition 2.5. *Let R be a locally finite ring. If R is abelian, then $R/J(R)$ is a strongly regular ring with $N(R) = J(R)$.*

Proof. Let $0 \neq a \in N(R)$ with $a^l = 0$ for some positive integer l. Assume that $ar \notin N(R)$ for some $r \in R$. Then $(ar)^s$ is a nonzero idempotent for some positive integer s by Lemma 2.4(1), say $(ar)^s = ab$ with $b \in R$. But it is central since R is abelian, and so $0 \neq (ar)^{s+1} = (ar)(ar)^s = (ar)((ar)^s)^l = (ar)(ab)^l = (a(ab)r)(ab)^{l-1} = (aa(ab)br)(ab)^{l-2} = \cdots = (a^l b^{l-1} r)(ab) = 0$, a contradiction. Thus ar is in $N(R)$ for all $r \in R$, similarly ra is also in $N(R)$ for all $r \in R$. Consequently aR and Ra are nil and so they are contained in $J(R)$; hence we have $N(R) \subseteq J(R)$. We also get $J(R) \subseteq N(R)$ since R is strongly π-regular by Lemma 2.4(2), showing $N(R) = J(R)$. It then follows that $R/J(R)$ is reduced and strongly π-regular. So Corollary 6 of Birkenmeier et al. [8] implies that every prime factor ring of $R/J(R)$ is a division ring; hence it is strongly regular by Theorems 1.21 and 3.2 of Goodearl [14]. $\square$

One may suspect $P(R) = J(R)$ in Proposition 2.5; however it does not hold generally by Example 2 in this paper. The converse of Proposition 2.5 is not true in general by the 2×2 upper triangular matrix ring over a locally finite strongly regular ring. It is well-known that locally finite domains are fields, but here we may get another proof as a consequence of above results.

Corollary 2.6. *Locally finite domains are fields; moreover every finite subset of a locally finite domain generates a finite field.*

Proof. Let R be a locally finite domain. Then R is a strongly regular ring by Proposition 2.5, so for any $0 \neq x \in R$ there is $y \in R$ with $x(1 - xy) = 0$. Since R is a domain, x is a unit and so R is a division ring. By Theorem 2.2(1), each finite subset of a locally finite ring generates a finite subring; hence any two elements $a, b \in R$ generate a finite domain, say D, which has the identity of R by Lemma 2.4(1). Consequently D is a field by the Wedderburn's well-known fact, proving $ab = ba$. Every finite subset of a locally finite domain generates a finite field similarly. $\square$

Note that algebraic closures of finite fields are locally finite but not finite. A ring is called *nonsingular* if it is both right and left nonsingular. Regular rings are nonsingular by Corollary 1.2 of Goodearl [14].

Theorem 2.7. *For a locally finite ring R, the following conditions are equivalent*:

(1) *R is an abelian semiprimitive ring.*
(2) *R is a reduced ring.*
(3) *R is a subdirect product of locally finite fields.*
(4) *R is a commutative regular ring.*
(5) *For each $0 \neq a \in R$, aR (or Ra) contains a nonzero central idempotent.*
(6) *R is a commutative nonsingular ring.*
(7) *R is a commutative semiprime ring.*

Proof. (1) $\Rightarrow$ (2): By Proposition 2.5.

(2) $\Rightarrow$ (3) : By Proposition 1.11 of Shin [26], every minimal prime ideal of R is completely prime since R is reduced. So R is a subdirect product of domains. But these domains are locally finite by Proposition 2.1(1), so they are fields by Corollary 2.6.

(3) $\Rightarrow$ (4): R is commutative reduced by the condition, and then R is commutative regular by Proposition 2.5.

(2) $\Rightarrow$ (5): Note that reduced rings are abelian. So given $0 \neq a \in R$, aR (Ra) contains a nonzero central idempotent by the reducedness of R and Lemma 2.4(1).

(5) $\Rightarrow$ (2): Assume that there exists $0 \neq a \in R$ with $a^n = 0$. But aR has a nonzero central idempotent by the condition, say ax; it then follows that $0 \neq ax = axax = a^2x^2 = a^2x^2ax = a^3x^3 = \cdots = a^nx^n = 0$, a contradiction. So R is reduced.

The proof of (4) $\Rightarrow$ (6) is obvious, while (6) $\Rightarrow$ (7) is proved by Chatters and Hajarnavis [12, Lemma 1.3].

(7) $\Rightarrow$ (1): Note that a commutative semiprime ring is reduced. R is strongly π-regular by Lemma 2.4(2), so $J(R)$ is nil and then $J(R) = 0$. □

In Theorem 2.7, the condition "R is an abelian semiprime ring" cannot be inserted by Example 2 in this paper.

Remark. Reduced locally finite rings are commutative regular by Theorem 2.7; hence the group of all units in R is abelian. For strongly regular rings, we may obtain a similar result as follows, using the method in the proof of Theorem 3.2 of Han [16]: Given a strongly regular ring R, we have that G is abelian if and only if R is commutative, where G is the group of all units in R.

Proposition 2.8. *For a right Goldie locally finite ring R, the following conditions are equivalent*:

(1) *R is an abelian semiprime ring.*
(2) *R is a strongly regular ring.*
(3) *R is a reduced ring.*
(4) *R is a finite direct product of locally finite fields.*
(5) *R is a commutative nonsingular ring in which every prime ideal is maximal.*
(6) *For each $0 \neq a \in R$, aR (or Ra) contains a nonzero central idempotent.*

Proof. (3) ⇔ (6) and (5) ⇒ (1) are proved by Theorem 2.7, so it suffices to show (1) ⇒ (4) since other directions are obvious. R is a semiprime right Goldie ring by the condition, moreover R and its classical right quotient ring coincide by Lemma 2.4(2); hence R is semisimple Artinian. But R is abelian by the condition, so R is a finite direct product of division rings. These are locally finite by Proposition 2.1, so they are locally finite fields by Corollary 2.6. □

The above condition "right Goldie" is not superfluous by Example 2 in this paper. As a byproduct of Proposition 2.8, we have the following: R is semiprime right Goldie if and only if R is a finite direct product of locally finite fields if and only if R is Artinian, when R is an abelian locally finite ring.

Proposition 2.9. *For a right Goldie locally finite ring R, the following conditions are equivalent*:

(1) *R is a semiprime ring.*
(2) *R is regular.*
(3) *R is a finite direct product of full matrix rings over locally finite fields.*
(4) *R is a semiprimitive ring in which every prime ideal is maximal.*

Proof. It suffices to show (1) ⇒ (3) since other directions are obvious. R is a finite direct product of full matrix rings over division rings by the proof of Proposition 2.8. These division rings are also locally finite fields also by the proof of Proposition 2.8. □

The above condition "right Goldie" is also not superfluous by Example 2 in this paper. As a byproduct of Proposition 2.9, we have the following: R is semiprime right Goldie if and only if R is a finite direct product of full matrix rings over locally finite fields if and only if R is Artinian, when R is a locally finite ring.

Given a locally finite ring R, we have concentrated on the case of $J(R) = N(R)$. Furthermore we now observe the case of $P(R) = N(R)$, recalling the result in Hirano [18, Theorem 1] and Birkenmeier et al. [8, Corollary 6] that for a ring R with $P(R) = N(R)$, R is strongly π-regular if and only if every prime factor ring of R is a division ring if and only if $R/P(R)$ is strongly regular with $P(R) = J(R) = N(R)$. Locally finite rings are strongly π-regular by Lemma 2.4(2), and so it may be somewhat meaningful to study relations between the local finiteness and the condition $P(R) = N(R)$. The following is only a restatement of Theorem 2.7 for the factor ring $R/P(R)$.

Proposition 2.10. *Let R be a locally finite ring. Then the following conditions are equivalent*:

(1) *$P(R) = N(R)$, i.e., $R/P(R)$ is reduced.*
(2) *Every prime factor ring of R is a locally finite field.*

(3) *$R/P(R)$ is a subdirect product of locally finite fields, and every prime ideal of R is maximal.*
(4) *$R/P(R)$ is commutative regular.*

In the preceding case we actually have $P(R)=N(R)=J(R)$ by Lemma 2.4(2). Due to Birkenmeier et al. [7], a ring R is called 2-*primal* if $P(R)=N(R)$; while Hirano [18] call such a ring an *N-ring*, observing the strong π-regularity of matrix rings over strong π-regular rings. It is well-known that the 2-primal condition helps to simplify other ring conditions, and the initial study of the 2-primal condition was done by Shin [26]. It is clear that a ring R is 2-primal if and only if $R/P(R)$ is reduced; hence commutative rings and reduced rings are 2-primal obviously. The *index of nilpotency* of a nilpotent element x in a ring R is the least positive integer n such that $x^n=0$. The *index of nilpotency* of a subset I of R is the supremum of the indices of nilpotency of all nilpotent elements in I. If such a supremum is finite, then I is said to be *of bounded index of nilpotency*.

Proposition 2.11. *Let R be a locally finite abelian ring. If R is of bounded index of nilpotency then R is a 2-primal ring with $P(R)=J(R)=N(R)$.*

Proof. If $N(R)=0$ then R is reduced so 2-primal. Suppose $N(R)\neq 0$ and let $0\neq a\in N(R)$. Then aR is a nonzero nil right ideal of R by the proof of Proposition 2.5. Since R is of bounded index of nilpotency, aR contains a nonzero nilpotent ideal of R by the Levitzki's lemma [17, Lemma 1.1]. $R/P(R)$ is also locally finite by Proposition 2.1(1), and it is an abelian ring of bounded index of nilpotency since $P(R)$ is nil. Assume that $P(R)\subsetneqq N(R)$, then we obtain a nonzero nilpotent ideal in $R/P(R)$, applying the above method again; that is a contradiction. Thus R is 2-primal, and $P(R)=J(R)=N(R)$ by Lemma 2.4(2). $\square$

Considering the conditions above, one may conjecture that locally finite abelian rings are 2-primal. However it does not hold in general by the following.

Example 2. There exists a locally finite abelian ring that is not 2-primal and not of bounded index of nilpotency. We construct such a ring in the same structure as the ring in Birkenmeier et al. [9, Example 3.3]. Let $\mathbb{V}$ be a vector space over $\mathbb{Z}_2$, the field of integers modulo 2, that is of countably infinite dimension with a basis $\{v(0),v(1),v(-1),\ldots,v(i),v(-i),\ldots\}$. Then there exists one and only one endomorphism $f(i)$ of $\mathbb{V}$, for $i=1,2,\ldots$ such that $f(i)(v(j))=0$ if $j\equiv 0\,(\mathrm{mod}\,2^i)$ and $f(i)(v(j))=v(j-1)$ if $j\not\equiv 0\,(\mathrm{mod}\,2^i)$. Denote F the ring, without identity, of endomorphisms of $\mathbb{V}$ generated by the endomorphisms $f(1),f(2),\ldots$. Now let R be the ring obtained from F by adjoining the identity map of $\mathbb{V}$. Then R is semiprime by the method in [6, p.540]. $R/F\cong\mathbb{Z}_2$ and $N(R)=J(R)=F\neq 0$ by the same argument as in Birkenmeier et al. [9, Example 3.3]. So R is not 2-primal. Notice that since R is local and $J(R)$ is nil, the identity is the only nonzero idempotent of R so R is abelian. Next let S be a finite subset of R and (S) be the multiplicative semigroup of

R generated by S. Let n be the largest one among the integers i such that $f(i)$ appears in S. Notice that $T^m = 0$ for some $m \leqslant 2^n$ and

$$T = \{\alpha \in F \mid \alpha \in (S) \quad \text{or} \quad 1 + \alpha \in (S)\};$$

hence T may be considered as a subset of the $m \times m$ full matrix ring $Mat_m(\mathbb{Z}_2)$ over $\mathbb{Z}_2$ since every $\alpha \in T$ is periodic and it is completely determined by the action on $v(j)$'s. So T is finite since $Mat_m(\mathbb{Z}_2)$ is finite, and consequently (S) is also finite because $\beta \in (S)$ with $\beta \notin T$, if any, is of the form $1 + \alpha$ with $\alpha \in T$. Therefore R is a locally finite abelian ring but not 2-primal; note that R is not of bounded index of nilpotency. □

Finite rings are clearly locally finite rings of bounded index of nilpotency; hence finite abelian rings must be 2-primal by Proposition 2.11. In this situation one may suspect that abelian rings are 2-primal and that finite rings are 2-primal. The former already does not hold in general by Example 2, and the latter is also not true in general by $n \times n$ full matrix rings over finite rings for $n \geqslant 2$.

A group is called *locally finite* if each finite subset generates a finite subgroup. For a group G and a ring R, denote the group ring of G over R by $R[G]$ and each element of it is expressed by $\sum rg$ with $r \in R$ and $g \in G$. A polynomial ring over any ring cannot be locally finite, but some factor rings of it may be locally finite as follows.

Proposition 2.12. (1) *Let R be a ring and G be a group. Then $R[G]$ is locally finite if and only if both R and G are locally finite.*

(2) *If R is a locally finite ring then so is $R[x]/x^nR[x]$ for all positive integer n, where $R[x]$ is the polynomial ring over R with x the indeterminate.*

Proof. (1) It suffices to show the sufficiency since the necessity is naturally obtained. Given a finite subset S of $R[G]$, let $A = \{r \in R \mid \sum rg \in S\}$ and $B = \{g \in G \mid \sum rg \in S\}$. Then A generates a finite subring of R by Theorem 2.2(1), say (A); while B generates a finite subgroup of G, say (B). Note that each element in the multiplicative semigroup generated by S, say (S), is of the form $\sum ah$ with $a \in (A)$ and $h \in (B)$. So (S) is also finite, implying that $R[G]$ is locally finite.

(2) The proof is obtained by (1). □

3. Semiperfect rings with nil Jacobson radical

Based on the examples in Cedó and Rowen [11] and Rowen [25], we will find some criteria on Jacobson radicals which assure that semiperfect rings may be strongly π-regular. In this situation, it is necessary to suppose that Jacobson radicals are nil since π-regular rings have nil Jacobson radicals. A ring R is called *local* if $R/J(R)$ is a division ring. Note that for a local ring with nil Jacobson radical each element is either invertible or nilpotent; hence we obtain the following.

Lemma 3.1. *A local ring with nil Jacobson radical is strongly π-regular.*

The condition "nil Jacobson radical" is not superfluous in Lemma 3.1 by a formal power series ring over a field. A ring R is called *semilocal* if $R/J(R)$ is semisimple Artinian, so any local ring is semilocal obviously. We call a ring R *semiperfect* if R is semilocal and idempotents modulo $J(R)$ can be lifted; one important case is when the Jacobson radical is nil. Cedó and Rowen [11, Example 1] found a semiperfect ring with locally nilpotent Jacobson radical but not strongly π-regular. So it is natural to consider the following condition.

Proposition 3.2. *For a ring R, suppose that $R/J(R)$ is a finite direct product of strongly π-regular rings and $J(R)$ is nil. If $J(R)$ is of bounded index of nilpotency then R is strongly π-regular.*

Proof. First note that a finite direct product of strongly π-regular rings are also strongly π-regular. $J(R)/P(R)$ is also of bounded index of nilpotency since $J(R)$ is a nil ideal of bounded index of nilpotency, so we have $J(R)=P(R)$ by the Levitzki's lemma [17, Lemma 1.1]. Then R is strongly π-regular by Proposition 3 of Hirano [18]. $\square$

The condition "of bounded index of nilpotency" in Proposition 3.2 is not superfluous by Example 1 of Cedó and Rowen [11] in fact the Jacobson radical in the example is locally nilpotent but the index of nilpotency is not bounded. One-sided Artinian rings are clearly strongly π-regular, so we have the following by Proposition 3.2.

Corollary 3.3. *Suppose that a ring R is semilocal and $J(R)$ is nil of bounded index of nilpotency. Then R is strongly π-regular.*

Remark. A subset of a ring is called *nilpotent* if some power of it is zero. Let R be a ring. Each case of the following has the property that nil ideals are nilpotent by Lanski [20], Lenagan [21] and Chatters and Hajarnavis [12, Theorem 1.34] respectively; hence we obtain strongly π-regular rings by Proposition 3.2 when $R/J(R)$ is a finite direct product of strongly π-regular rings and $J(R)$ is nil:

(i) R is a ring with right Krull dimension (in the sense of Gabriel and Rentschler, see [15]),
(ii) R is right Goldie, and
(iii) R satisfies the ascending chain condition on both right and left annihilators.

A ring R is called a right *p.p.* ring if each principal right ideal of R is projective. It is simply checked that a ring R is right p.p. if and only if the right annihilator of each element of R is generated by an idempotent. Domains, semisimple Artinian rings, and $n \times n$ upper triangular matrix rings over division rings are right p.p. rings, where $n=1,2,\ldots$. A ring R is said to have *enough idempotents* if the identity can be written as the sum of a finite number of orthogonal primitive idempotents. A ring has enough idempotents if it has no infinite sets of orthogonal idempotents.

Corollary 3.4. *Let R be a semilocal ring with $J(R)$ nil. If R is a right p.p. ring then R is strongly π-regular.*

Proof. Since R is semilocal and $J(R)$ is nil, R is semiperfect by Proposition 3.6.1 of Lambek [19] and moreover R has enough idempotents by Proposition 3.6.4 of Lambek [19]. Then $J(R)$ is nilpotent by Lemma 8.6 of Chatters and Hajarnavis [12] implying that R is strongly π-regular by Corollary 3.3. □

Next we study the strong π-regularity or π-regularity of more generalized situations of Lemma 3.1.

Proposition 3.5. *Let R be a ring with nontrivial central idempotent. If $R/J(R)$ is a direct product of two division rings and $J(R)$ is nil then R is strongly π-regular.*

Proof. Let e be a nontrivial central idempotent in R. Then e must be a primitive idempotent by the condition that $R/J(R)$ is a direct product of two division rings. Note that R is semiperfect since $J(R)$ is nil, so $eR=eRe$ is a local ring by Proposition 3.7.2 of Lambek [19]. Similarly $fR=fRf$ is also a local ring with $f=1-e$. Consequently R is a direct product of two local rings with nil Jacobson radicals through $R=eR\oplus fR$, and thus R is a direct product of two strongly π-regular rings by Lemma 3.1. Therefore R is strongly π-regular. □

A ring R is called *right continuous* if (1) every right ideal of R is essential in a right direct summand of R and (2) every right ideal of R isomorphic to a right direct summand is generated by an idempotent. Right self-injective rings are right continuous by Theorem 4.7 of Utumi [27]. If R is a right continuous ring then $R/J(R)$ is regular by Theorem 4.6 of Utumi [27], so it may be interesting to study conditions for which right continuous rings may be strongly π-regular.

Proposition 3.6. *Let R be a right continuous ring with $J(R)$ nil. If R is 2-primal then R is strongly π-regular.*

Proof. $R/J(R)$ is regular by Theorem 4.6 of Utumi [27] since R is right continuous; and so R is strongly π-regular by Theorem 1 of Hirano [18]. □

Proposition 3.7. *Let R be a right continuous ring with $J(R)$ nil. If $J(R)$ is not essential in R as a right ideal and $R/J(R)$ is a direct product of two division rings, then R is strongly π-regular.*

Proof. Let $R/J(R)=D_1\oplus D_2$ with D_i division rings, and $x\in R$. If x is either nilpotent or invertible then we are done, so we assume that x is neither nilpotent nor invertible. Then $\bar{x}=x+J(R)=(a,0)$ or $\bar{x}=x+J(R)=(0,b)$, for some $0\neq a\in D_1$ and $0\neq b\in D_2$. Say $\bar{x}=x+J(R)=(a,0)$. Write $\bar{R}=R/J(R)$. Clearly $(1,0)\in\bar{x}\bar{R}$, and there exists a nonzero idempotent e of R such that $e+J(R)=(1,0)$ by Proposition 3.6.1 of Lambek

[19]. The condition $R/J(R) = D_1 \oplus D_2$ implies that e is a primitive idempotent; moreover R is semiperfect and so eRe is a local ring by Proposition 3.7.2 of Lambek [19]. Now assume that $eJ(R) \neq 0$ and $(1-e)J(R) \neq 0$; then since R is right continuous, there exist nonzero idempotents $f, g \in R$ such that $eJ(R)$ and $(1-e)J(R)$ are essential in fR and gR respectively. Consequently $J(R) = eJ(R) + (1-e)J(R)$ is essential in $fR + gR$; but $f + J(R) = (1, 0)$ and $g + J(R) = (0, 1)$ since $R/J(R) = D_1 \oplus D_2$, so we get $fR + gR = R$. Then $J(R)$ is essential in R, a contradiction to the given condition. Therefore $eJ(R) = 0$ or $(1-e)J(R) = 0$. Say $eJ(R) = 0$, then eR is a minimal right ideal of R by Corollary 17.20 of Anderson and Fuller [1] because eRe is local. Thus $(1-e)R$ is a maximal right ideal of R and it is furthermore a maximal ideal of R because $R/J(R) = D_1 \oplus D_2$. It follows that $R(1-e) \subseteq (1-e)R$ and $eR(1-e) = 0$. So $er = ere$ for all $r \in R$; especially $ex = exe$ and hence we have $x = ex + (1-e)x = exe + (1-e)x$ and $(1-e)x \in J(R)$, implying that $x^n \in Re$ (then $x^n = x^n e$) for some positive integer n. Since eRe is local and $ex^n e \notin J(R)$, there exists $y = eye \in eRe$ such that $ex^n ey = yex^n e = e$; hence

$$x^n = x^n e = x^n ex^n ey = x^n x^n y = x^{2n} y.$$

The proofs of other cases are similar. R is strongly π-regular by these arguments. □

Lemma 3.8. *For a ring R, suppose that $R/J(R)$ is a finite direct product of division rings. If $J(R)$ is nil and $x \notin J(R)$, then xR (Rx) contains a nonzero idempotent.*

Proof. Set $R/J(R) = D_1 \oplus D_2 \oplus \cdots \oplus D_k$ with D_i division rings, and denote $r + J(R)$ by $\bar{r}$ for $r \in R$. If xR contains $J(R)$ then the proof is simple, so we assume that x is noninvertible. Then $\bar{x} = (a_1, a_2, \ldots, a_k)$ with $a_i \in D_i$ and $\emptyset \subsetneqq \{i \mid a_i = 0\} \subsetneqq \{1, 2, \ldots, k\}$. Here we may suppose $a_1 \neq 0$, then there exists $y \in R$ such that $\bar{x}\bar{y} = \bar{y}\bar{x} = (1, b_2, \ldots, b_{k-1}, 0)$ with each nonzero b_i (if any) 1. So $(\bar{x}\bar{y})^l = \bar{x}\bar{y}$ for every positive integer l, especially $xy - (xy)^2 \in J(R)$. $(xy - (xy)^2)^n = 0$ for some positive integer n since $J(R)$ is nil. Set $z = xy$, then we have

$$0 = (z - z^2)^n = z^n - \left(n1 - \frac{n(n-1)}{2}z + \cdots - (-1)^n z^{n-1}\right) z^{n+1}.$$

Set $a = n1 - n(n-1)/2z + \cdots - (-1)^n z^{n-1}$, then $z^n = az^{n+1}$ and inductively we obtain $z^n = a^m z^{m+n}$ for every positive integer m. Now let $u = (az)^n$, then $u^2 = (az)^{2n} = a^n(a^n z^{n+n}) = a^n z^n = u$ because $az = za$. Note that $\bar{u} = \bar{z} = (1, b_2, \ldots, b_{k-1}, 0)$, and so u is a nonzero idempotent of R. But $u \in xR$ by the definition of z. The argument for Rx is similar. □

A ring R is called an *exchange ring* [28] if R_R has the exchange property. A right continuous ring is exchange by Proposition 1.6 of Nicholson [22]. A ring I without identity is called an *exchange ring* [3, definition after Theorem 11.2] if for each $x \in I$ there exist an idempotent $e \in I$ and $r, s \in I$ such that $e = xr = x + s - xs$.

Proposition 3.9. *Let R be a right continuous ring with $J(R)$ nil. If every essential right ideal of R is 2-sided and $R/J(R)$ is a direct product of two division rings, then R is strongly π-regular.*

Proof. Let $x \in R$ and $R/J(R) = D_1 \oplus D_2$ with D_i division rings. Write $\bar{R} = R/J(R)$ and $\bar{r} = r + J(R)$. If x is either nilpotent or invertible then we are done, so we assume that x is neither nilpotent nor invertible. Then $\bar{x} = x + J(R) = (a, 0)$ or $\bar{x} = x + J(R) = (0, b)$, for some $0 \neq a \in D_1$ and $0 \neq b \in D_2$. Say $\bar{x} = x + J(R) = (a, 0)$.

First consider the case that xR is not essential in R. There exists an idempotent e of R such that xR is essential in eR because R is right continuous; hence $x = ex = exe + ex(1 - e)$. Note that $e + J(R) = (1, 0)$ because xR is not essential in R, and so $exe \notin J(R)$. Also e must be a primitive idempotent in view of $R/J(R) = D_1 \oplus D_2$; hence R is semiperfect by Proposition 3.6.1 of Lambek [19] and eRe is local by Proposition 3.7.2 of Lambek [19]. Then there exists $y = eye \in eRe$ such that $exey = yexe = e$, and thus

$$x = ex = e(exexy^2)x = x^2(y^2x).$$

Next consider the case that xR is essential in R, then xR is 2-sided by hypothesis. Assume that $\bar{x} = (a, 0)$ with $a \neq 0$. There exists $y \in R$ such that $\bar{x}\bar{y} = \bar{y}\bar{x} = (1, 0)$. Then xy^2x is another preimage of $(1, 0)$. Since R is an exchange ring, xR is an exchange ring by Example (1) of Ara [3]. Then Rx is an exchange ring by Proposition 1.3 of Ara et al. [4]. So there are $tx \in Rx$ and $sx \in Rx$ such that $h := xy^2x(tx) = xy^2x + sx - xy^2x(sx)$. Then $\bar{h} = (1, 0)$, and so $0 \neq h = h^2 \in xRx$. Now since xR is a 2-sided ideal of R, $h \in xRx \subseteq xxR$ and so $h = x^2a$ for some $a \in R$. Set $h_1 = xahx$, then clearly $0 \neq h_1$ and we have that $h_1^2 = xahxxahx = xahhhx = xahx = h_1$ and $hx = hhx = xxahx = xh_1$. Consequently, xRx contains nonzero idempotents h and h_1 with $hx = xh_1$.

Let $h_0 = h$, then by preceding arguments we may find nonzero idempotents h_k for $k = 1, 2, \ldots$ such that $h_k \in xRx$ and $h_{k-1}x = xh_k$. Let $v_k = 1 - h_k + h_{k-1}$ for $k = 1, 2, \ldots$. Then $v_kx = xv_{k+1}$, and v_k is invertible because $\bar{v}_k = \overline{1 - h_k} + \overline{h_{k-1}} = (0, 1) + (1, 0) = (1, 1)$ and $J(R)$ is nil. Since $(1 - h_1)x$ is in $J(R)$ and $v_1x = (1 - h_1 + h)x = (1 - h_1)x + hx = (1 - h_1)x + xh_1$, it follows that $(v_1x)^n = ((1 - h_1)x + xh_1)^n \in Rh_1$ (so $(v_1x)^n = (v_1x)^nh_1$) for some positive integer n. Notice that h_1 is also a primitive idempotent of R because $\bar{h}_1 = (1, 0)$, h_1Rh_1 is local by Proposition 3.7.2 of Lambek [19] since R is semiperfect. Since $h_1(v_1x)^nh_1 \notin J(R)$, there exists $y = h_1yh_1 \in h_1Rh_1$ such that $h_1(v_1x)^nh_1y = yh_1(v_1x)^nh_1 = h_1$. Then

$$(v_1x)^n y(v_1x)^n = (v_1x)^n yh_1(v_1x)^nh_1 = (v_1x)^nh_1 = (v_1x)^n$$

and

$$(v_1x)^n = (v_1x)(v_1x)\cdots(v_1x) = (v_1x)(v_1x)\cdots(v_1x)(xv_2)$$
$$= (v_1x)(v_1x)\cdots(v_1x)(xxv_3v_2) = \cdots = x^n(v_{n+1}v_n\cdots v_2).$$

This implies $x^n(v_{n+1}v_n\cdots v_2)yx^n(v_{n+1}v_n\cdots v_2) = x^n(v_{n+1}v_n\cdots v_2)$. But $v_{n+1}v_n\cdots v_2$ is invertible and so we obtain $x^nzx^n = x^n$, where $z = (v_{n+1}v_n\cdots v_2)y$. Since xR is 2-sided, $Rx^n \subseteq x^nR$ and so $zx^n = x^nz'$ for some $z' \in R$. Hence $x^n = x^{n+1}t$ for some $t \in R$. Therefore R is strongly π-regular by the preceding two arguments. □

There exists a right continuous ring R such that every essential right ideal is 2-sided and $R/J(R)$ is a direct product of two division rings but R is not π-regular as follows. Given a ring R and a bimodule ${}_RM_R$, the *trivial extension* of R by M is the ring $R \oplus M$

with the usual addition and the following multiplication: $(r_1,m_1)(r_2,m_2)=(r_1r_2,r_1m_2+m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r\in R$ and $m\in M$ and the usual matrix operations are used.

Example 3. For a prime number p, let $\hat{\mathbb{Z}}_{(p)}$ be the completion of the localization $\mathbb{Z}_{(p)}$, i.e., $\hat{\mathbb{Z}}_{(p)}$ is the ring of p-adic integers, where $\mathbb{Z}$ is the integers. Set $\mathbb{Z}_{p^\infty}$ be the Prüfer p-group. Then by Osofsky [23], the trivial extension S of $\hat{\mathbb{Z}}_{(p)}$ by $\mathbb{Z}_{p^\infty}$ is a commutative self-injective (hence continuous) ring. Let $R=S\oplus S$, then R is also a commutative self-injective ring. Note that $J(R)=J(S)\oplus J(S)$ with $J(S)=\left\{\begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \mid a\in p\hat{\mathbb{Z}}_{(p)} \text{ and } m\in\mathbb{Z}_{p^\infty}\right\}$. So $R/J(R)=\mathbb{Z}_p\oplus\mathbb{Z}_p$, a direct sum of fields, where $\mathbb{Z}_p$ is the ring of integers modulo p. However $J(R)$ is not nil, implying that R is not π-regular. □

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